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EXAM II, MTH 320, Fall 2016

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QUESTION 1. Let D be a group with 55 elements.

(i) **(6 points)**. Convince me that D is not simple.

Solution: We know that D has an element of order 11, and hence D has a subgroup, say H, with 11 elements. Since [D : H] = 5 and 5 is the smallest prime factor of 55, we know that H must be normal. Thus D is not simple.

(ii) **(8 points)**. Assume that D has a normal subgroup, say H, such that $|H| = 5$. Prove that D is cyclic.

Solution: Let K be a normal subgroup of D with 5 elements and let H as in (i). We know HK is a subgroup of D. Thus $|HK| = 5$ or 11 or 55. Since K and H are subgroups of HK, we conclude that $|HK| = 55$. Thus $HK = D$. It is clear that $H \cap K = \{e\}$. Hence by one of the results in class, we have $D/(H \cap K) \simeq D/H \times D/K$ and thus $D \simeq D/H \times D/K$. Since $|D/H| = 5$ and $|D/K| = 11$, we conclude that $D/H \simeq Z_5$ and $D/K \simeq Z_{11}$. Thus $D \simeq Z_5 \times Z_{11} \simeq Z_{55}$ is cyclic.

QUESTION 2. (8 points). Given that H is a normal subgroup of a group $(D, *)$ such that $|H| = 11$. Assume that $D/H =< a * H >$ (i.e., D/H is cyclic and generated by $a * H$) for some $a \in D \setminus H$ such that $a * h = h * a$ for every $h \in H$. Prove that D is abelian

Solution: I wrote this question to see how many of you read the proof I give in CLASS. Similar proof to if $D/C(D)$ is cyclic, then D is abelian. Here we go: Let $x, y \in D$. Show $x * y = y * x$. Hence $x = a^i * H, y = a^k * H$ in D/H. Thus $x = a^i * b, y = a^k * c$ for some $b, c \in H$. Now since $|H| = 11$, H is cyclic and hence abelian. Thus $b * c = c * b$. Also by hypothesis, we have $a * b = b * a$ and $a * c = c * a$. Hence $x * y = a^{i+k} * b * c = a^{i+k} * c * b = y * x$.

QUESTION 3. (6 points). Let $F : Z_{15} \rightarrow Z_{12}$ be a nontrivial group homomorphism. Find $Ker(F)$ and $Image(F)$.

Solution: We know $Z_{15}/Ker(F) \simeq Image(F)$. Hence by staring (and keep in mind that Image(F) is a subgroup of Z_{12} and $|image(F)|$ must be a factor of the two numbers 12 and 15), we conclude that $|Z_{15}/Ker(F)| = |Image(F)| = 3$. Thus $Image(F) = \{0, 4, 8\}$, and in order that $|Z_{15}/Ker(F)| = 3$ we must have $|Ker(F)| = 5$. Thus $Ker(F) = \{0, 3, 6, 9, 12\}$.

QUESTION 4. (6 points). Let $F : Z \rightarrow Z_{20}$ be a nontrivial group homomorphism. Given that F is not ONTO (not surjective) and $5 \in Image(F)$. Find $Ker(F)$ and $Image(F)$.

Solution: Since F is not onto and $5 \in Image(F)$, $< 5 >= \{0, 5, 10, 15\}$ is the only subgroup of Z_{20} that is not equal to Z_{20} and contains 5. Thus $Image(F) = \{0, 5, 10, 15\}$. We know every subgroup of Z is of the form kZ. Hence $Z/Ker(F) = Z/kZ \simeq Image(F) = \{0, 5, 10, 15\} \simeq Z_4$. Thus $K = 4$. Hence $Ker(F) = 4Z$.

QUESTION 5. (6 points). Let D be an abelian group with p^3 elements for some prime integer p. Assume that D has a unique subgroup of order p. Prove that D is cyclic.

Solution: We Know that (1) $D \simeq Z_{p^3}$ or (2) $D \simeq Z_p \times Z_{p^2}$ or (3) $D \simeq Z_p \times Z_p \times Z_p$. If D is isomorphic to the groups in (2) or (3), then clearly D has more than one subgroup with p elements. Thus $D \simeq Z_{p^3}$ is cyclic.

QUESTION 6. (6 points). Let D be a a noncyclic abelian group with 32 elements. Assume that $|a| = 16$ for some $a \in D$. Up to isomorphism, find all such groups.

Solution: We know (1) $D \simeq Z_{32}$ or (2) $D \simeq Z_2 \times Z_{16}$ or (3) $D \simeq Z_{k_1} \times \cdots Z_{k_m}$ where $k_1, ..., k_m \in \{2, 4, 8\}$. Now D is not isomorphic to Z_{32} since D is not cyclic. D is not isomorphic to a group as in (3) since all such groups have elements of order 8 or less. Thus $D \simeq Z_2 \times Z_{16}$.

QUESTION 7. (6 points). Assume that a group D has unique subgroup H where $|H| = 2016$. Prove that H is a normal subgroup of D.

Solution: Let $a \in D$. Show $a * H = H * a$. Since $C_a(H) = a * H * a^{-1}$ is a subgroup od D with cardinality equals to the cardinality of H, we conclude $a * H * a^{-1} = H$. Thus $a * H = H * a$.

QUESTION 8. (i) **(5 points)**. Is $U(27) \simeq Z_{18}$? explain

(ii) **(5 points)**. Is $(1\ 2\ 4)o(1\ 3) \in A_4$? explain

(iii) **(5 points)**. Is every abelian group with 45 elements isomorphic to $Z_{15} \times Z_3$? explain

(iv) **(5 points)**. Let $a = (1\ 3\ 4\ 5)o(2\ 4\ 1)$. Find $|a|$

(v) **(5 points)**. Let $a \in S_7$ and $m = |a|$. What is the maximum value of m. Explain briefly.

Solution: (i-iv): all of you got it right. For (v): just observe that a must be written as disjoint cycles say $a = a_1\ o\ a_2\ o \cdots o\ a_k$ and $|a|$ = LCM[length of a_1, length of a_2, ..., length a_k] = m = maximum. Now it should be clear that for m to be maximum $k = 2$, $|a_1| = 4$ and $|a_2| = 3$. Hence $m = 12$.

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